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Stochastic Modelling

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Abstract
This lecture consists of two parts: first part is a tribute to and reminiscences of J K Ghosh; second part is description of a well-known probabilistic model leading to the plausibility of Bose-Einstein condensation.

Key words: Distribution of balls; Bose-Einstein statistics, Bayes interpretation, Boltzmann principle.

1 (a) JKG and His Work

When I heard on September 30, 2017 that Professor Jayanta Kumar Ghosh (JKG) passed away in Purdue, it was a sad moment for me. Yes, he was ailing for some weeks; yes, he was in a nursing home; yes, he made me prepared for the news; but yet when the actual moment arrived it was indeed very sad. He supervised my doctoral work and my thoughts went back to several years in the past. The only thing that we can do is to remember him and learn from his life.

I shall repeat briefly, for completeness, a few points Professor Bikas Sinha mentioned already.

JKG was born on May 23, 1937 in Calcutta (now Kolkata) to Sri Ambar Nath Ghosh and Srimati Shanti Lata Ghosh. His father was a non-practising lawyer. He received doctoral degree, for work done under the supervision of Professor H K Nandi, from Calcutta University. After visiting University of Illinois, Urbana-Champagne he returned to India and joined as faculty of Indian Statistical Institute, subsequently becoming its Director. After retiring from ISI he spent at Purdue University as Professor of Statistics, visiting India and ISI annually.

His doctoral work in the early sixties relates to the average sample number and other aspects (J. K. Ghosh: *Calcutta Statist. Assoc. Bull. 1960, vol 9, p 139-144 and 1961, vol 10, p. 73-92 and 1964, vol 13, p. 101-122.*) of the Sequential Probability Ratio Test of Abraham Wald.

I thank Professor Vinod Gupta, President, Society of Statistics, Computer and Applications, for his kind invitation to participate in the J K Ghosh memorial session of IPECS 2018 conference at Pondicherry.
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In 1965, he made an important contribution to Invariance and Sufficiency (Hall, Wijsman and Ghosh: *Ann. Math. Statist. Volume 36, Number 2 (1965), p. 575-614*). In a parametric decision problem where underlying family of probabilities is invariant under a group of transformations on the sample space; in which order should we reduce the problem of inference: Use principle of invariance first and then principle of sufficiency or the principle of sufficiency first and then the principle of invariance.

Starting early seventies he contributed, among other things, to second order asymptotics (J K Ghosh and Kasala Subrahmanyam: *Sankhya A 1974, vol 36, p. 325-358*); a concept introduced by C R Rao; a procedure that provides an effective measure to choose an estimator with the best possible summary of data for drawing inference.

In 1978 he made, jointly with Rabi Bhattacharya, fundamental contribution to the validity of Edgeworth expansions (R N Bhattacharya and J K Ghosh: *Ann. Statist. 1976, vol 6 p. 434-451*), settling an old conjecture of Wallace.

He was very much interested in applied work and analysis of data. He collaborated with Supriya Sengupta and B S Mazumder on aspects of sedimentation and size distribution of suspended particles. He was very proud of the flume project he set up at the ISI where experiments were conducted and data collected.

He published, jointly with Bikas Sinha and T J Rao on history of Statistics in India. He was interested in philosophy and participated in (also contributed to) conference on Consciousness at Ramakrishna Mutt, Belur, Howrah. He contributed to several other areas like survival analysis, statistical quality control and so on.

He was more and more drawn towards Bayesian methods and had become a full-fledged Bayesian in later years; a master and authority on the subject. He contributed extensively to several aspects: Bayes factors; Model Selection; Bernardo Priors; consistency, normality and convergence rates of posterior distributions; and so on. Many of these are in collaboration with Jim Berger, R V Ramamoorthi, Subhasish Ghoshal and others.

He edited several books and conference Proceedings. He authored several books: *Invariance in Testing and Estimation, 1967,* an Indian Statistical Institute monograph; *Higher Order Asymptotics, 1994,* an Institute of Mathematical Statistics/American Statistical Association monograph; *Bayesian Nonparametrics, 2003,* a Springer monograph jointly with R V Ramamoorthi; *An Introduction to Bayesian Analysis - Theory and Methods, 2006,* a Springer monograph jointly with Mohan Delampady and Tapas Samanta.

He supervised a large number of students for their doctoral dissertations, both at the Indian Statistical Institute and at Purdue University. While at the Indian Statistical Institute, he was a teacher in high demand by the students. Apart from research and teaching, he had his share of administrative responsibilities too which he discharged diligently and with understanding. He was Director of the Indian Statistical Institute; Editor of ‘Sankhya’ for several years.

JKG was Fellow of several societies including: Indian National Science Academy; Indian Academy of Sciences; Institute of Mathematical Statistics; Institute of Statistical Mathematics. He was President of the Statistical Section of the Indian Science Congress(1991); President of the International Statistical Institute (1993). He was recipient of several awards, including: Shanti Swarup Bhatnagar Prize of the CSIR, India (1981); Mahalanobis gold medal of the Indian Science Congress (1998), P.V. Sukhatme prize of the Government of India (2000), Honorary D.Sc. by the B.C. Roy Agricultural University in West Bengal, India (2006), the Lifetime Achievement award of the International Indian Statistical Association (2010), Honorary D.Sc by the Indian Statistical Institute (2012), Padma Shree from the Government of India (2014).

He had a sense of humour. After he visited Japan, some one in the Tea club asked him if he can read Japanese. He promptly answered, Yes; adding a little later, if it is written in English.

He had immense love for Calcutta and passionate about Tagore's Poetry. His fondness and attachment to ISI was enormous — so much so that he was unhappy and disappointed when things did not go the way he wanted.

As Anirban Dasgupta says: *In a clearly distinguished career, Professor Ghosh influenced the work of various spectra of statisticians at different times, was an undisputed leader of Indian statistics, and produced an enviable number of successful students. It is an extraordinary legacy.*

JKG said in one of his letters to Ramamoorthi: *As one gets old and the once solid foundations of one's life begin to disintegrate, a warm friendship matters more than anything else.*

When I heard that Mrs Ghosh passed away on September 17, 2017, I was deeply saddened and my thoughts went back fifty years when young JKG entered my office to invite for his wedding reception. When I heard about JKG few days later I was left wondering: did she leave early to make arrangements for his comfortable stay *'there'*.

They have a son Indraneel Ghosh, currently Professor of Chemistry at University of Arizona and a daughter Joyee Ghosh, currently faculty in the Statistics department at University of Iowa.

1 (b) JKG and Some More Academics

In 1966, a little after joining the ISI, JKG gave a series of lectures on construction of Haar measure; something that he used in his discussion on invariance. He always wished to be thorough with the mathematics he needed.

In 1967, when J L Doob visited ISI, he was in-charge of the academic proceedings. That was one of the well-planned visits. It was known that Doob would lecture on Potential theory and Brownian motion. C R Rao constituted a potential theory study group to read, discuss and acquaint ourselves with these topics before the arrival of Doob. While Doob was lecturing, I would make notes for each lecture and distribute at the beginning of next lecture. During that one month, I spent several hours discussing with JKG. He was very patient, helpful and clarified many points.

In 1968 he wrote an article on Marginal Sufficiency, RTS (Research and Training School) Tech Report No. 54 (1968). Here is the abstract:

Suppose that $X_1, \cdots, X_n$ *are independently distributed random variables and* $T(X_1, ..., X_n)$ *is a statistic sufficient for each* X_i*. Is* T *sufficient for* $(X_1, \cdots, X_n)$*? In other words, does marginal sufficiency imply sufficiency if* $X_1, \cdots, X_n$ *are independent? The rather surprising answer to this question (given in Theorem 1) is yes, at least in the dominated case. (Of course, the answer is no if the* X'*s are dependent.)*

The problem arose in an unpublished paper of Dr. V. S. Huzurbazar who conjectured the correct answer.

It was a very difficult paper to understand. Several years later R R Bahadur presented, in a lecture at ISI, Kolkata, an understandable proof in a special case. Subsequently, I made notes, in full generality, of this theorem. None of these was ever published.

In the meanwhile, the Russian statistician V N Sudakov Published; *The marginal sufficiency of statistics*, Zap. Nauchn. Sem. LOMI, 1972, Volume 29, p. 92 - 101. This is what he says:

Several years ago the Indian statistician V. S. Huzurbazar advanced the hypothesis that in the case of a repeated sample, the marginal sufficiency of a statistic implies its sufficiency. The proof of this hypothesis was announced in a preprint published by Ghosh in 1968, but my colleagues and I, all specialists in mathematical statistics, have run into considerable difficulties in attempting to reconstruct the complete proof from Ghosh's preprint. We have not even been able to extricate a reasonably clear idea that might be of help in arriving at the required proof. Below we set forth the proof $\cdots$.

Many years later, in 1986, the Japanese statistician Hirokichi Kudo also published his version: *On Marginal Sufficiency*; Statistics and Decisions vol 4, p. 301-320.

2 (a) Distribution of Balls

In this second part of the lecture, we shall discuss a probabilistic model for distributing balls into boxes and its importance. Suppose we have N balls and k boxes. These balls have to be distributed in the k boxes.

(i) Let us first assume that the balls are distinguishable. This is called the Maxwell-Boltzmann experiment.

Now Total number of configurations equals k^N, simply because we can place each ball in any one of the boxes and balls being distinguishable, different placements lead to distinguishable arrangements. We assume that these configurations are equally likely.

Given $(N_1, \cdots, N_k)$ where each N_i is a non-negative integer and $\sum N_i = N$ the number of configurations such that box i has N_i balls (for all i) equals

$$\frac{N!}{N_1! N_2! \cdots N_k!}$$

Thus the configuration $(N_1, \cdots, N_k)$ has chance

$$\frac{N!}{N_1! N_2! \cdots N_k!} \frac{1}{k^N}$$

(ii) Let us now assume that the balls all look alike. This is called Bose-Einstein experiment.

How many distinct configurations can our eye perceive? Remember, the balls being alike it does not make sense to ask which ball went into box one. For instance, how many arrangements are there so that box 1 gets one ball and box 2 gets $(N-1)$ balls? There is only one such. More generally given $(N_1, \cdots, N_k)$ where each N_i is a non-negative integer such that $\sum N_i = N$ the number of configurations such that box i has N_i balls (for each i) equals *exactly one*.

We assume that these configurations are equally likely. How many total configurations are there?

$$\binom{k+N-1}{k-1}$$

Thus each configuration $(N_1, \cdots, N_k)$ has chance

$$\frac{1}{\binom{k+N-1}{k-1}}.$$

Before passing, we make two observations. Firstly, this formula has a Bayesian explanation, as Sudhakar Kunte (*Sankhya [1977], vol. 39, p.305-308*) found out. Suppose you have a die with k faces, face i appearing has chance p_i. Here $p_i > 0$ for each i and $\sum p_i = 1$. Suppose you roll the die and put the ball in the box shown on the die; do it for each ball. Then the distribution of the balls is the so called multinomial distribution. More precisely, following earlier notation, the configuration $(N_1, N_2, \cdots, N_k)$ has chance

$$\frac{N!}{N_1! N_2! \cdots N_k!} p_1^{N_1} p_2^{N_2} \cdots p_k^{N_k}.$$

Let Δ be the set of all vectors $(p_1, \cdots, p_{k-1})$ with each $p_i > 0$ and $\sum p_i < 1$; with the understanding $p_k = 1 - \sum\limits_1^{k-1} p_i$. Imagine now that we have one die for each $(p_1, \cdots, p_{k-1})$. We pick a die at random and then use that die to distribute all the balls as described above. Then what is the chance of the configuration $(N_1, \cdots, N_k)$? It is clearly

$$\frac{1}{|\Delta|} \int_\Delta \frac{N!}{N_1! N_2! \cdots N_k!} p_1^{N_1} p_2^{N_2} \cdots p_{k-1}^{N_{k-1}} \left(1 - \sum_1^{k-1} p_i\right)^{N_k} dp_1 \cdots dp_{k-1} = \frac{1}{\binom{k+N-1}{k-1}}.$$

Here $|\Delta| = [(k-1)!]^{-1}$ is volume of Δ.

Secondly, the suggestion that all the distinguishable outcomes are equally likely is not as innocent as it looks like. Suppose you have a box with 100 balls one green and all others red. Assume that all red balls look alike. Pick a ball at random. There are only two distinguishable outcomes: red ball and green ball. Can you believe that these are equaly likely? The above suggestion does not apply in this situation.

The reason for putting forward such a suggestion is the following. Imagine balls are actually photons and boxes are energy levels. You are trying to model the distribution of the photons in various energy levels. Observations confirm that photons obey this!

2 (b) Boltzmann Principle

The key principle of statistical mechanics, Boltzmann principle, is the following. Suppose that a system in (thermal) equilibrium can be in one of m states

$$1, 2, \cdots, m \quad \text{with energies} \quad E_1, E_2, \cdots, E_m$$

then the probability that the system is in state i is proportional to

$$\exp\{-\beta E_i\}.$$

More precisely, the chances that the system is in state i equals $\frac{1}{Q}e^{-\beta E_i}$ where $Q = \sum\limits_n e^{-\beta E_n}$, called the partition function. In these expressions the constant β (which does not concern us) equals $(\kappa T)^{-1}$ where κ is the Boltzmann constant, T is temperature.

As Feynman puts, this fundamental law (formulated in the quantum set up) is the summit of statistical mechanics and the entire subject is either slide down from the summit as the principle is applied to various cases, or the climb up where the fundamental law is derived and the concepts of thermal equilibrium and temperature T clarified. We shall apply this principle now.

2 (c) Boltzmann Principle and Distribution of Balls

Let us return to the setup of $\mathbf{2(a)}$. We have a system of N particles and k energy levels: $E_1, \cdots, E_k$. Each particle can be in any of the levels. Let us now consider the N-particle system. We assume that energies add up. Thus a configuration $(N_1, \cdots, N_k)$ for the system has energy $\sum N_i E_i$.

(i) Maxwell Boltzmann set up:

For the N-particle system any one specific configuration $(N_1, \cdots, N_k)$ has chance

$$c^{-1}\exp\{-\beta \sum N_i E_i\} = c^{-1}\Pi a_i^{N_i} \quad a_i = \exp\{-\beta E_i\}$$

where

$$c = \sum_{k^N \text{ config.}} \Pi a_i^{N_i} = (a_1 + \cdots + a_k)^N$$

Thus probability of each specific $(N_1, \cdots, N_k)$ configuration equals

$$\Pi \frac{a_i^{N_i}}{(a_1 + \cdots + a_k)^N} = \Pi p_i^{N_i} \qquad\qquad p_i = \frac{a_i}{\sum a_j}$$

In other words we have multinomial probabilities $(N;\ p_1, \cdots, p_k)$.

Expected number of particles in energy box E_i is Np_i. If Number of particles becomes large, then each Box too has a large number of particles.

(ii) Bose-Einstein set-up:

Einstein asked: Bose's idea + Boltzmann principle =?

Let us arrange the energy levels in increasing order:

$$E; \quad E + x_1; \quad E + x_2; \cdots\cdots; E + x_{k-1}$$

where $0 < x_1 < x_2 < \cdots < x_{k-1}$. Accordingly, with the obvious understanding, we denote configurations as $(N_0, N_1, \cdots, N_{k-1})$ instead of $(N_1, \cdots, N_k)$. Energy of the system in this configuration equals $\sum N_i E_i = NE + \sum_1^{k-1} N_i x_i$

Probability of this configuration is proportional to

$$e^{-\beta[NE + \sum_1^{k-1} N_i x_i]} = e^{-\beta NE} \prod_1^{k-1} z_i^{N_i}$$

where $z_i = \exp[-\beta x_i]$ for $1 \leq i \leq k-1$. Note that $0 < z_i < 1$ because $x_i > 0$.

Probability of the configuration $(N_0, N_1, \cdots, N_{k-1})$ equals

$$\frac{\prod_1^{k-1} z_i^{N_i}}{\sum_{\text{config.}} \prod_1^{k-1} z_i^{N_i}}$$

expected number of particles in Energy Level $E + x_1$ equals

$$\langle N_1 \rangle = \frac{\sum_{\text{config}} N_1 \prod_1^{k-1} z_i^{N_i}}{\sum_{\text{config.}} \prod_1^{k-1} z_i^{N_i}} = \frac{\sum_{m \leq N} m\, z_1^m \sum_{??} \prod_2^{k-1} z_i^{N_i}}{\sum_{\text{config.}} \prod_1^{k-1} z_i^{N_i}}$$

Here $??$ is the set of all $(N_2, \cdots, N_{k-1})$ with sum at most $N - m$. Denote,

$$f(N, m) = \sum_{\sum_2^{k-1} N_i \leq N - m} \prod_2^{k-1} z_i^{N_i}; \qquad f(N) = \sum_{\text{config.}} \prod_1^{k-1} z_i^{N_i}.$$

Thus

$$\langle N_1 \rangle = \frac{\sum\limits_{m \le N} m z_1^m \, f(N,m)}{f(N)}$$

$$\lim_{N\to\infty} f(N) = \prod_1^{k-1} \frac{1}{(1-z_i)}; \quad \lim_{N\to\infty} f(N,m) = \prod_2^{k-1} \frac{1}{(1-z_i)}$$

whatever be m. Now DCT applies and we have

$$\lim_{N\to\infty} \langle N_1 \rangle = \frac{z_1/(1-z_1)^2}{1/(1-z_1)} = \frac{z_1}{(1-z_1)}$$

Thus

$$\lim_{N\to\infty} \langle \sum_1^{k-1} N_i \rangle = \sum_1^{k-1} \frac{z_i}{(1-z_i)}$$

which is a finite number.

Let us denote the lowest energy state E as 'ground state' and the other states $E + x_i$ $(1 \le i \le k-1)$ as excited states. The above observation can be summarized as follows, leading to the plausibility of Bose-Einstein Condensation.

if number of particles is large, then Most of them are in Ground State and Only finitely many are in the excited states.

2 (d) Remarks

We conclude with three remarks.

(i) Bose-Einstein Statistics, namely distribution of balls considering them as indistinguishable, is the invention of S N Bose (1924). When the referee, unable to deviate from conventional wisdom, did not accept the result for publication, Bose sent it to Einstein with a letter:

Respected Sir, I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. $\cdots\cdots$ *if you think the paper worth publication I shall be grateful if you arrange for its publication in Zeitschrift fur Physik.*

Though a complete stranger to you, I do not hesitate in making such a request. Because we are all your pupils though profiting from your teachings through your writings $\cdots\cdots$.

Then Einstein himself translated and submitted for publication with the comment:

In my opinion Bose's derivation of the Planck formula signifies an important advance. The method used also yields the quantum theory of the ideal gas, as I will work out in detail elsewhere.

JV Narlikar says: *S. N Bose's work on particle statistics which clarified the behaviour of photons and opened the door to new ideas on statistics of microscopic systems that obey the rules of quantum theory was one of the top ten achievements of twentieth century Indian science and could* *be considered in the Nobel prize class.*

(ii) As mentioned already, the above argument is only a plausibility and not rigorous. It is not enough to let number of particles increase, one should also control total energy of the system; (but then make energy states well-separated making beta large and hence) temperature should go to zero. Also one should study the fluctuations instead of just expectations. There is a vast literature on the subject at an advanced level and in various settings; see especially articles of P. Ferrari, C. Landim and V.Sisko; or V. Maslov and V. Nazaikinskii or Lieb or Sourav Chatterjee and Persi Diaconis; and others.

(iii) The purpose of the above discussion is purely pedagogic: to expose the student to a beautiful idea in a simple setting; to encourage him to learn the mathematics needed for a realistic (and correct) setting. In our curriculum, most of the examples in elementary probability courses are tossing coin, rolling die, picking from deck of playing cards etc. Even if we throw balls into boxes, it is regarded as a combinatorial exercise with unclear intentions. The student is left wondering whether probability is mere fun or a useful endeavour, till he takes up advanced courses.