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# On Zero-inflated Generalized Alternative Hyper-Poisson Distribution and its Properties

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**Abstract**

A generalized version of the zero-inflated alternative hyper-Poisson distribution of Kumar and Ramachandran (*Statistica*, 2021) is introduced and study some of its important statistical properties such as mean, variance, recursion relations for probabilities, raw moments and factorial moments. The estimation of the parameters of this distribution is considered and the distribution has been fitted to a well-known data set. Further a generalized likelihood ratio test procedure is applied for testing the significance of the inflation parameter.

*Key words:* Confluent hypergeometric series; Count data modeling; Generalized likelihood ratio test; Model selection; Simulation; Zero-inflated Hermite distribution.

**AMS Subject Classifications:** 62K05, 05B05

## 1. Introduction

Bardwell and Crow (1964) considered a generalized version of the Poisson family of distributions through the following probability mass function (p.m.f.), for x= 0, 1, 2, ..., $\lambda > 0$ and $\theta > 0$.

$$f(x) = P(X = x) = \frac{1}{\phi(1;\lambda;\theta)} \frac{\theta^x}{(\lambda)_x}, \tag{1}$$

where

$$\phi(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}$$

is the confluent hypergeometric series, in which

$(a)_k = a(a+1)(a+2)...(a+k-1) = \dfrac{\Gamma(a+k)}{\Gamma(a)}$, for k=1, 2, ... and $(a)_0$=1. The distribution with p.m.f. (1) is known in the literature as the hyper-Poisson distribution (HPD).

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Kumar and Nair (2012) considered an alternative form of the hyper-Poisson distribution (AHPD). The p.m.f.. of the AHPD is the following, for y=0, 1, 2, ... .

$$P(Y=y)=\frac{\gamma^y}{(\rho)_y}\phi(1+y;\rho+y;-\gamma), \tag{2}$$

in which $\gamma>0$ and $\rho>0$. The Poisson distribution is the special case of the AHPD when $\rho=1$. Moreover over dispersion and under dispersion in cases of $\rho>1$ and $\rho<1$ is also one of the important characteristics of the AHPD. Kumar and Ramachandran (2021) introduced a zero-inflated version of the alternative hyper-Poisson distribution (ZIAHPD) whose p.m.f.. is given by

$$f(z)=\begin{cases}\omega+(1-\omega)\phi(1;\rho;-\gamma), \ z=0\\ (1-\omega)\frac{\gamma^z}{(\rho)_z}\phi(1+z;\rho+z;-\gamma), \ z=1, \ 2, \ ... ,\end{cases} \tag{3}$$

in which $\omega\in[0,1]$, $\rho>0$ and $\gamma>0$. When $\rho=1$, the ZIAHPD reduces to the zero-inflated Poisson distribution.

Through this paper we develop further a generalized version of the zero-inflated alternative hyper-Poisson distribution (ZIAHPD) of Kumar and Ramachandran (2021) which we call “the zero-inflated generalized alternative hyper-Poisson distribution (ZIGAHPD)” and discuss some of its important statistical properties. In section 2, we present the definition of the ZIGAHPD and obtain its probability generating function, expressions for its mean and variance, and recursion formulae for probabilities, raw moments and factorial moments. Further, the estimation of the parameters of the model is discussed in section 3 and a test procedure is discussed in section 4. In section 5 both procedures discussed in sections 3 and 4 are illustrated with its relevence with the help of a real life data set.

We need the following series representations in the sequel.

$$\sum_{x=0}^{\infty}\sum_{r=0}^{\infty}A(r,x)=\sum_{x=0}^{\infty}\sum_{r=0}^{x}A(r,x-r) \tag{4}$$

$$\sum_{x=0}^{\infty}\sum_{r=0}^{\infty}A(r,x)=\sum_{x=0}^{\infty}\sum_{r=0}^{[\frac{x}{m}]}A(r,x-rm). \tag{5}$$

## 2. Definition and Properties

We present the definition of the ZIGAHPD and discuss some of its properties.

**Definition 1:** A discrete random variable $M$ is said to follow “the zero-inflated generalized alternative hyper-Poisson distribution or in short ZIGAHPD” with parameters $\omega$, $\lambda$, $\theta_1$, $\theta_2$ and $\theta_3$ if its p.m.f. is

$$\begin{aligned} f(m) &= P(M=m) \qquad (6)\\ &= \begin{cases}\omega+(1-\omega)\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)], & m=0\\ (1-\omega)\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}}\phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]\frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^k}{k!}\frac{\theta_3^j}{j!}, & m=1,2,...\\ 0, otherwise\end{cases}\end{aligned}$$

in which $\omega \in [0, 1)$, $\lambda > 0$, $\theta_1 > 0$, $\theta_2 \geq 0$ and $\theta_3 \geq 0$.

Important special cases of the ZIGAHPD includes the following cases.

1. when $\omega = 0$, ZIGAHPD $\rightarrow$ generalized alternative hyper-Poisson distribution (GAHPD) of Kumar and Sandeep (2022).
2. when $\theta_2 = \theta_3 = 0$, ZIGAHPD $\rightarrow$ the ZIAHPD of Kumar and Ramachandran (2021) with p.m.f.. (3).
3. when $\theta_2 = \theta_3 = 0$ and $\lambda = 1$, ZIGAHPD $\rightarrow$ ZIPD of Lambert (1992).
4. when $\theta_3 = 0$ and $\lambda = 1$, ZIGAHPD $\rightarrow$ zero-inflated Hermite distribution (ZIHD) of Kumar and Ramachandran (2020).
5. when $\omega = 0$, $\theta_3 = 0$ and $\lambda = 1$, ZIGAHPD $\rightarrow$ Hermite distribution (HD) of Kemp and Kemp (1965).
6. when $\omega = 0$ and $\theta_3 = 0$, ZIGAHPD $\rightarrow$ modified alternative hyper-Poisson distribution (MAHPD) of Kumar and Nair (2013).
7. when $\omega = 0$, $\theta_2 = 0$ and $\theta_3 = 0$, ZIGAHPD $\rightarrow$ alternative hyper-Poisson distribution (AHPD) of Kumar and Nair (2012).

Now we obtain the following results.

**Result 1:** The probability generating function (p.g.f) G(t) of the ZIGAHPD with p.m.f. (6) is the following.

$$G(t) = \omega + (1 - \omega)\ \phi[1; \lambda; \theta_1(t - 1) + \theta_2(t^2 - 1) + \theta_3(t^3 - 1)]. \tag{7}$$

**Proof:** By definition, the p.g.f of the ZIGAHPD having p.m.f. (6) is given by

$$\begin{aligned} G(t) &= \sum_{m=0}^{\infty} f(m) t^m \\ &= \omega + (1-\omega)\phi[1; \lambda; -(\theta_1 + \theta_2 + \theta_3)] + (1-\omega) \sum_{m=1}^{\infty} \sum_{j=0}^{[\frac{m}{3}]} \sum_{k=0}^{[\frac{m}{2}]} t^m \frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}} \\ &\times \ \phi[1 + m - 2j - k; \lambda + m - 2j - k + z; -(\theta_1 + \theta_2 + \theta_3)] \frac{\theta_1^{m-3j-2k} \theta_2^k \theta_3^j}{(m - 3j - 2k)! k! j!} \\ &= \omega + (1-\omega) \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{3}]} \sum_{k=0}^{[\frac{m}{2}]} t^m \frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}} \frac{\theta_1^{m-3j-2k} \theta_2^k \theta_3^j}{(m - 3j - 2k)! k! j!} \\ &\times \ \phi[1 + m - 2j - k; \lambda + m - 2j - k + z; -(\theta_1 + \theta_2 + \theta_3)]. \end{aligned}$$

In the light of the following result $(\lambda)_x(\lambda+x)_r=(\lambda)_{x+r}$, we have

$$\begin{aligned}
G(t) &= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(m-2j-k)!(m-3j-k)!}{(m-3j-k)!j!k!(m-3j-2k)}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}} \qquad (8)\\
&\times\ \phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]t^m\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}}\\
&\times\ \phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]t^m\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}}t^m\\
&\times\ \sum_{r=0}^{\infty}\frac{(1+m-2j-k)_r}{(\lambda+m-2j-k)_r}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{r!}\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}}t^m\\
&\times\ \frac{(1)_{m-2j-k}}{(m-2j-k)!(\lambda)_{m-2j-k}}\sum_{r=0}^{\infty}\frac{(1+m-2j-k)_r}{(\lambda+m-2j-k)_r}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{r!}\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}(m-2j-k)!}t^m\\
&\times\ \sum_{r=0}^{\infty}(1)_{m-2j-k}\frac{(1+m-2j-k)_r}{(\lambda+m-2j-k)_r}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{r!}\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\theta_1^{m-3j-2k}\theta_2^k\theta_3^j t^m\\
&\times\ \sum_{r=0}^{\infty}(1)_{m-2j-k+r}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{(\lambda)_{m-2j-k+r}r!(m-2j-k)!}\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\theta_1^{m-3j-2k}\theta_2^k\theta_3^j t^m\\
&\times\ \sum_{r=0}^{\infty}\binom{m-2j-k+r}{r}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{(\lambda)_{m-2j-k+r}}\\
&= \omega+(1-\omega)\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2j-k+r}{r}\binom{m-2j-k}{j}\binom{m-3j-k}{k}\\
&\times\ \frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k+r}}[-(\theta_1+\theta_2+\theta_3)]^r t^m.
\end{aligned}$$

Using inequality (4), we obtain

$$\begin{aligned}
G(t) &= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]}\binom{m+2j+r-k}{r}\binom{m+j-k}{j}\binom{m-k}{k} \qquad (9)\\
&\times \frac{(\theta_1 t)^{m-2k}(\theta_2 t^2)^k(\theta_3 t^3)^j}{(\lambda)_{m+r+j-k}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+j+r-k}{r}\binom{m+j+k}{j}\binom{m+k}{k}\\
&\times \frac{(\theta_1 t)^{m}(\theta_2 t^2)^k(\theta_3 t^3)^j}{(\lambda)_{m+r+j+k}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m}\binom{m+j+r}{r}\binom{m+j}{j}\binom{m}{k}\\
&\times \frac{(\theta_1 t)^{m-k}(\theta_2 t^2)^k(\theta_3 t^3)^j}{(\lambda)_{m+r+j}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+j+r}{r}\binom{m+j}{j}\frac{(\theta_1 t+\theta_2 t^2)^m(\theta_3 t^3)^j}{(\lambda)_{m+r+j}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\binom{m+r}{r}\binom{m}{j}\frac{(\theta_1 t+\theta_2 t^2)^{m-j}(\theta_3 t^3)^j}{(\lambda)_{m+r+j}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+r}{r}\frac{(\theta_1 t+\theta_2 t^2+\theta_3 t^3)^m}{(\lambda)_{m+r}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{m}{r}\frac{(\theta_1 t+\theta_2 t^2+\theta_3 t^3)^{m-r}}{(\lambda)_{m}}[-(\theta_1+\theta_2+\theta_3)]^r\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\frac{[\theta_1 t+\theta_2 t^2+\theta_3 t^3-(\theta_1+\theta_2+\theta_3)]^m}{(\lambda)_m}\\
&= \omega + (1-\omega)\sum_{m=0}^{\infty}\frac{[\theta_1 (t-1)+\theta_2 (t^2-1)+\theta_3 (t^3-1)]^m}{(\lambda)_m}
\end{aligned}$$

which on simplification gives (7). □

**Result 2:** The mean and variance of the ZIGAHPD with p.g.f (7) are

$$Mean = \frac{(1-\omega)}{\lambda}(\theta_1+2\theta_2+3\theta_3)$$

and

$$Variance = \left\{\left(\frac{2}{\lambda+1}-\frac{1-\omega}{\lambda}\right)(\theta_1+2\theta_2+3\theta_3)^2+(\theta_1+4\theta_2+9\theta_3)\right\}\frac{(1-\omega)}{\lambda}.$$

Next we derive certain recursion formulae for the probabilities, raw moments and factorial moments of the ZIGAHPD through the following results. Hereafter, for the convenience of the notation, we write $f_m(\lambda^{(j)})$ for the probability mass function $f(m)$ as given in (6) where $\lambda^{(j)} = (1+j, \lambda+j)$ for $j = 0, 1, 2, \ldots$ .

**Result 3:** A simple recursion formula for probabilities $f_m(\lambda^{(j)})$ of the ZIGAHPD is the following

$$f_1(\lambda^{(0)}) = \frac{\theta_1}{\lambda}\left(f_0(\lambda^{(1)}) - \omega\right) \tag{10}$$

and

$$(m+1)f_{m+1}(\lambda^{(0)}) = \frac{1}{\lambda}\left[\theta_1 f_m(\lambda^{(1)}) + 2\theta_2 f_{m-1}(\lambda^{(1)}) + 3\theta_3 f_{m-2}(\lambda^{(1)})\right], \ for\ m \geq 1. \tag{11}$$

**Proof:** The p.g.f of the ZIGAHPD given in (7) can be written as

$$\begin{aligned} G(t) &= \omega + (1-\omega)\ \phi[1;\lambda;\theta_1(t-1)+\theta_2(t^2-1)+\theta_3(t^3-1)] \\ &= \sum_{m=0}^{\infty} t^m f_m(\lambda^{(j)}). \end{aligned} \tag{12}$$

On differentiating (12) with respect to t, we obtain the following.

$$\sum_{m=0}^{\infty}(m+1)f_{m+1}(\lambda^{(0)})t^m = \frac{(1-\omega)}{\lambda}(\theta_1+2\theta_2 t+3\theta_3 t^2)\ \phi[2;\lambda+1;\theta_1(t-1)+\theta_2(t^2-1)+\theta_3(t^3-1)]. \tag{13}$$

Also from (12), we have

$$(1-\omega)\phi[2;\lambda+1;\theta_1(t-1)+\theta_2(t^2-1)+\theta_3(t^3-1)] = \sum_{m=0}^{\infty} f_m(\lambda^{(1)})t^m - \omega. \tag{14}$$

Combining relations (13) and (14) we obtain

$$\sum_{m=0}^{\infty}(m+1)f_{m+1}(\lambda^{(0)})t^m = \frac{(\theta_1+2\theta_2 t+3\theta_3 t^2)}{\lambda}\left[\sum_{m=0}^{\infty} f_m(\lambda^{(1)})t^m - \omega\right]. \tag{15}$$

Now, on equating the coefficients of $t^0$ on both sides of (15), we get (10), and on equating the coefficients of $t^y$ on both sides of (15), we get (11). □

**Result 4:** For $r \geq 0$, a recursion formula for raw moments $\mu_r(\lambda^{(0)})$ of the ZIGAHPD is

$$\mu_{[r+1]}(\lambda^{(0)}) = \frac{1}{\lambda}\sum_{k=0}^{r}\binom{r}{k}(\theta_1 + 2^{k+1}\theta_2 + 3^{k+1}\theta_3)\mu_{[r-k]}(\lambda^{(1)}). \tag{16}$$

**Proof:** For any $t \in \Re = (-\infty,\infty)$ and $i = \sqrt{-1}$, the characteristic function of the ZIGAHPD is

$$\begin{aligned} H(t) &= G(e^{it}) \\ &= \omega + (1-\omega)\ \phi[1;\lambda;\theta_1(e^{it}-1)+\theta_2(e^{2it}-1)+\theta_3(e^{3it}-1)] \\ &= \sum_{r=0}^{\infty}\mu_r(\lambda^{(0)})\frac{(it)^r}{r!}. \end{aligned} \tag{17}$$

Differentiating (17) with respect to $t$, we get

$$\sum_{r=0}^{\infty}\mu_{[r+1]}(\lambda^{(0)})\frac{(it)^r}{r!}=\frac{(1-\omega)}{\lambda}\left(\theta_1e^{it}+2\theta_2e^{2it}+3\theta_3e^{3it}\right)\times \\ \phi[2;\lambda+1;\theta_1(e^{it}-1)+\theta_2(e^{2it}-1)+\theta_3(e^{3it}-1)]. \quad (18)$$

Also, from (17) we have

$$(1-\omega)\phi[2;\lambda+1;\theta_1(e^{it}-1)+\theta_2(e^{2it}-1)+\theta_3(e^{3it}-1)]=\sum_{r=0}^{\infty}\mu_{[r]}(\lambda^{(1)})\frac{(it)^r}{r!}-\omega. \quad (19)$$

Combining (18) and (19), we get

$$\sum_{r=0}^{\infty}\mu_{[r+1]}(\lambda^{(0)})\frac{(it)^r}{r!}=\frac{1}{\lambda}\left(\theta_1e^{it}+2\theta_2e^{2it}+3\theta_3e^{3it}\right)\left(\sum_{r=0}^{\infty}\mu_{[r]}(\lambda^{(1)})\frac{(it)^r}{r!}-\omega\right). \quad (20)$$

On expanding the exponential functions in the right hand side expression of (20) and equating the coefficients of $\frac{(it)^r}{r!}$ on both sides we get (16). □

**Result 5:** For $r\geq 1$ a simple recursion formula for factorial moments $\mu_{[r]}(\lambda^{(0)})$ of the ZIGAHPD is

$$\mu_{[r+1]}(\lambda^{(0)})=\frac{\theta_1}{\lambda}\mu_{[r]}(\lambda^{(1)})+\frac{2\theta_2r}{\lambda}\mu_{[r-1]}(\lambda^{(1)})+\frac{3\theta_3r(r-1)}{\lambda}\mu_{[r-2]}(\lambda^{(1)}). \quad (21)$$

**Proof:** The factorial moment generating function F(t) of the ZIGAHPD with p.g.f. (7) is the following

$$\begin{aligned} F(t) &= G(1+t) \\ &= \omega+(1-\omega)\phi[1;\lambda;\theta_1t+\theta_2\{(1+t)^2-1\}+\theta_3\{(1+t)^3-1\}] \\ &= \sum_{r=0}^{\infty}\mu_{[r]}(\lambda^{(0)})\frac{t^r}{r!}. \end{aligned} \quad (22)$$

On differentiating (22) with respect to t, we get

$$\frac{1}{\lambda}(1-\omega)(\theta_1t+\theta_2\{(1+t)^2-1\})+\theta_3\{(1+t)^3-1)\}\times \\ \phi[2;\lambda+1;\theta_1t+\theta_2\{(1+t)^2-1\}+\theta_3\{(1+t)^3-1\}]=\sum_{r=0}^{\infty}\mu_{[r+1]}(\lambda^{(0)})\frac{t^r}{r!}. \quad (23)$$

From (22), we obtain

$$(1-\omega)\phi[2;\lambda+1;\theta_1t+\theta_2\{(1+t)^2-1\}+\theta_3\{(1+t)^3-1\}]=\sum_{r=0}^{\infty}\mu_{[r]}(\lambda^{(1)})\frac{t^r}{r!}-\omega. \quad (24)$$

Equations (23) and (24) together lead to

$$\sum_{r=0}^{\infty} \mu_{[r+1]}(\lambda^{(0)}) \frac{t^r}{r!} = \frac{\theta_1 + 2\theta_2(1+t) + 3\theta_3(1+t)^2}{\lambda} \left( \sum_{r=0}^{\infty} \mu_{[r]}(\lambda^{(1)}) \frac{t^r}{r!} - \omega \right). \tag{25}$$

On equating the coefficients of $\frac{t^r}{r!}$, we get (21). □

## 3. Maximum likelihood estimation

Here we consider the estimation of the parameters $\omega$, $\lambda$, $\theta_1$, $\theta_2$ and $\theta_3$ of the ZIGAHPD by the method of maximum likelihood. For any $m = 0, 1, 2, ...$, let $A(m)$ be the observed frequency of $m$ events and let $z$ be the highest value of $m$ observed. Then the likelihood function of the sample is given by

$$L(\Theta; m) = \prod_{m=0}^{z} [f(m)]^{A(m)},$$

where $f(m)$ is the p.m.f. of the ZIGAHPD given in (6).
Now $L(\Theta; m)$ can be written as

$$L(\Theta; m) = (f(0))^s \prod_{m=1}^{z} (f(m))^{A(m)},$$

where $s = A(0)$.

Then the log-likelihood function can be written as

$$\begin{aligned} \ln L(\theta; m) &= s \ln [\omega + (1-\omega)\phi(1; \lambda; \theta_1 + \theta_2 + \theta_3)] + \sum_{m=1}^{z} A(m) \\ &\times \ln \left[ (1-\omega) \sum_{j=0}^{[\frac{m}{3}]} \sum_{k=0}^{[\frac{m}{2}]} \frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}} \phi[1+m-2j-k; \lambda+m-2j-k; -(\theta_1+\theta_2+\theta_3)] \right. \\ &\times \left. \frac{\theta_1^{m-3j-2k}\ \theta_2^k\ \theta_3^j}{(m-3j-2k)!\ k!\ j!} \right] \end{aligned} \tag{26}$$

Assume that $\hat{\omega}$, $\hat{\lambda}$, $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ be the maximum likelihood estimators of the parameters $\omega$, $\lambda$, $\theta_1$, $\theta_2$ and $\hat{\theta}_3$ of the ZIGAHPD. Now, on differentiating the log-likelihood function (26) with respect to $\omega$, $\lambda$, $\theta_1$, $\theta_2$ and $\theta_3$ and equating to zero, we obtain the following likelihood equations:

$$\frac{\partial \ln L}{\partial \omega} = 0$$

which implies

$$\frac{s\,[1 - \phi[1; \lambda; -(\theta_1+\theta_2+\theta_3)]]}{\omega + (1-\omega)\phi[1; \lambda; -(\theta_1+\theta_2+\theta_3)]} - \sum_{m=1}^{z} \frac{A(m)}{(1-\omega)} = 0, \tag{27}$$

$$\frac{\partial \ln L}{\partial \lambda} = 0$$

which implies

$$\frac{s(1-\omega)}{\omega+(1-\omega)\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]}\sum_{r=0}^{\infty}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{(\lambda)_r}[\psi(\lambda)-\psi(\lambda+r)] \tag{28}$$
$$+\sum_{m=1}^{z}A(m)\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{[(\lambda)_{m-2j-k}]^2}\frac{\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(m-3j-2k)!k!j!}$$
$$\times\Bigg\{\frac{1}{(\lambda)_{m-2j-k}}\sum_{r=0}^{\infty}\frac{[-(\theta_1+\theta_2+\theta_3)]^r}{r!(\lambda+m-2j-k)_r}(1+m-2j-k)_r\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]$$
$$\times[\psi(\lambda+m-2j-k)-\psi(\lambda+m-2j-k+r)]-\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]$$
$$\times\frac{1}{(\lambda)_{m-2j-k}}[\psi(\lambda+m-2j-k)-\psi(\lambda+m-2j-k+r)]\Bigg\}=0,$$

$$\frac{\partial \ln L}{\partial \theta_1} = 0$$

which implies

$$\frac{-s{(1-\omega)}/{\lambda}\phi[2;\lambda+1;-(\theta_1+\theta_2+\theta_3)]}{\omega+(1-\omega)\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]}+\sum_{m=0}^{z}A(m)\frac{1}{\xi}\Bigg\{\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}} \tag{29}$$
$$\times\quad\frac{\theta_1^{m-3j-2k-1}}{(m-3j-2k-1)!}\frac{\theta_2^k\theta_3^j}{k!j!}\phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]$$
$$-\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}}\frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^k\theta_3^j}{k!j!}\frac{1}{\lambda+m-2j-k}$$
$$\times\,\phi[2+m-2j-k;\lambda+1+m-2j-k;-(\theta_1+\theta_2+\theta_3)]\Big\}=0,$$

$$\frac{\partial \ln L}{\partial \theta_2} = 0$$

which implies

$$\frac{-s{(1-\omega)}/{\lambda}\phi[2;\lambda+1;-(\theta_1+\theta_2+\theta_3)]}{\omega+(1-\omega)\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]}+\sum_{m=0}^{z}A(m)\frac{1}{\xi}\Bigg\{\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}} \tag{30}$$
$$\times\quad\frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^{k-1}\theta_3^j}{(k-1)!j!}\phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]$$
$$-\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}}\frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^k\theta_3^j}{k!j!}\frac{1}{\lambda+m-2j-k}$$
$$\times\,\phi[2+m-2j-k;\lambda+1+m-2j-k;-(\theta_1+\theta_2+\theta_3)]\Big\}=0$$

and

$$\frac{\partial \ln L}{\partial \theta_3} = 0$$

which implies

$$\frac{-s^{(1-\omega)}/_{\lambda}\phi[2;\lambda+1;-(\theta_1+\theta_2+\theta_3)]}{\omega+(1-\omega)\phi[1;\lambda;-(\theta_1+\theta_2+\theta_3)]} + \sum_{m=0}^{z} A(m)\frac{1}{\xi}\left\{\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}}\right. \quad (31)$$
$$\times \quad \frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^k\theta_3^{j-1}}{k!(j-1)!}\phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)]$$
$$-\sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}}{(\lambda)_{m-2j-k}}\frac{\theta_1^{m-3j-2k}}{(m-3j-2k)!}\frac{\theta_2^k\theta_3^j}{k!j!}\frac{1}{\lambda+m-2j-k}$$
$$\left.\times\ \phi[2+m-2j-k;\lambda+1+m-2j-k;-(\theta_1+\theta_2+\theta_3)]\right\},$$

in which $\psi(\lambda) = \frac{\partial}{\partial\lambda} log\ \Gamma(\lambda)$ and
$\xi = \sum_{j=0}^{[\frac{m}{3}]}\sum_{k=0}^{[\frac{m}{2}]}\frac{(1)_{m-2j-k}\theta_1^{m-3j-2k}\theta_2^k\theta_3^j}{(\lambda)_{m-2j-k}(m-3j-2k)!j!k!}\phi[1+m-2j-k;\lambda+m-2j-k;-(\theta_1+\theta_2+\theta_3)].$

On solving the likelihood equations (27), (28), (29), (30) and (31) with the help of some mathematical softwares, say *Mathematica*, one can obtain the maximum likelihood estimators of the parameters of the proposed distribution.

## 4. Testing

In order to test the significance of the inflation parameter $\omega$ of the ZIGAHPD, we adopt the following generalized likelihood ratio test (GLRT) procedure. Here the null hypothesis is

$$H_0 : \omega = 0 \ \textit{against the alternative hypothesis}\ H_1 : \omega \neq 0.$$

The test statistic suggested in the case of GLRT is given by

$$-2\ln\psi = 2\left(\iota_1 - \iota_2\right), \quad (32)$$

where, $\iota_1 = \ln L(\hat{\theta}; m)$, where $\hat{\theta}$ is the maximum likelihood estimator for $\theta = (\omega, \lambda, \theta_1, \theta_2, \theta_3)$ with no restrictions, and $\iota_2 = lnL(\hat{\theta}^*; m)$, in which $\hat{\theta}^*$ is the maximum likelihood estimator for $\theta$ under the null hypothesis $H_0$. The test statistic defined in (32) is asymptotically distributed as $\chi^2$ with one degree of freedom.

## 5. Applications

In this section we illustrate all the procedures discussed in sections 3 and 4 with the help of a real life data set.

The data here considered is a biological data based on the distribution of European Corn borer Pyrausta Naubilalis in field corn (Avi *et al.* (2008)). We have fitted the ZIGAHPD to the data set and considered the fitting of the models - ZIAHPD, ZIHD, ZIPD, ZIMAHPD and GAHPD for comparison. For comparing the models we computed the values of $\chi^2$, AIC, BIC and AICc. The numerical results obtained are presented in Tables 1. Based on the computed values of $\chi^2$, AIC, BIC and AICc as presented in Table 1, one can observe that the ZIGAHPD gives a better fit to the data set while all other models such as ZIAHPD, ZIHD, ZIPD, ZIMAHPD and GAHPD are not appropriate.

We have also calulated the values of the test statistic. The value of the test statistic for $\ln L(\hat{\theta}^*; m) = -169.1$ and $\ln L(\hat{\theta}; m) = -144.3$ is given by 49.6. The critical value of the test having 5% level of significance and degree of freedom one is 3.84, so that the null hypothesis is rejected in all the cases. Thus, we conclude that the additional parameter $\omega$ in the model is significant.

**Table 1: Distribution of the spread of European Corn borer Pyrausta Naubilalis in field corn (Rodriguez et.al., 2008) and the expected frequencies computed using ZIAHPD, ZIHD, ZIPD, ZIMAHPD, GAHPD and ZIGAHPD.**

| *Count* | *Observed frequency* | *ZIAHPD* | *ZIHD* | *ZIPD* | *ZIMAHPD* | *GAHPD* | *ZIGAHPD* |
|---|---|---|---|---|---|---|---|
| 0 | 206 | 265.73 | 200.4 | 252.15 | 213.65 | 245.4 | 200.3 |
| 1 | 143 | 148.4 | 100.515 | 151.43 | 112.51 | 157.6 | 142.6 |
| 2 | 128 | 144.2 | 137.73 | 140.6 | 179.2 | 151.4 | 119.8 |
| 3 | 107 | 127.1 | 110.6 | 118.35 | 119.6 | 128.048 | 100.8 |
| 4 | 71 | 80.361 | 90.7 | 75.78 | 93.2 | 73.42 | 80.4 |
| 5 | 36 | 7.29 | 59.8 | 23.9 | 35.3 | 8.5 | 38.5 |
| 6 | 32 | 4.6 | 38.6 | 7.1 | 19.08 | 7.4 | 39.4 |
| 7 | 17 | 2.3 | 21.3 | 5.4 | 5.9 | 3.6 | 12.6 |
| 8 | 14 | 1.25 | 11.5 | 4.02 | 2.58 | 2.45 | 19.2 |
| 9 | 7 | 0.5 | 5.7 | 1.25 | 0.67 | 1.98 | 5.2 |
| 10 | 7 | 0.25 | 2.7 | 1.6 | 0.23 | 0.87 | 7.9 |
| 11 | 2 | 0.0024 | 1.52 | 0.0015 | 0.05 | 0.50 | 1.2 |
| 12 | 3 | 0.01 | 0.61 | 0.35 | 0.018 | 0.23 | 1.8 |
| 13 | 3 | 0.006 | 0.2 | 0.021 | 0.004 | 0.20 | 2.3 |
| 14 | 1 | 0.00003 | 0.08 | 0.006 | 0.00815 | 0.35 | 2.2 |
| 15 | 1 | 0.00009 | 0.03 | 0.0002 | 0.00025 | 0.05 | 1.1 |
| 16 | 1 | 0.000006 | 0.011 | 0.00004 | 0.00007 | 0.002 | 2.7 |
| 17 | 2 | 0.000007 | 0.003 | 0.041 | 0.000013 | 0.00035 | 2.5 |
| 18 | 1 | 0.000009 | 0.0013 | 0.000008 | 0.0000021 | 0.000007 | 1.5 |
| *Total* | 782 | 782 | 782 | 782 | 782 | 782 | 782 |
| *df* | | 3 | 7 | 6 | 3 | 3 | 6 |
| *Estimates* | | $\lambda$=12.09<br>$\omega$=0.59<br>$\theta$=7.0009 | $\lambda$=1.11<br>$\omega$=0.14<br>$\theta$=0.89 | $\lambda$=3.95<br>$\omega$=0.17 | $\lambda$=0.1<br>$\omega$=0.8<br>$\theta_1$=0.32<br>$\theta_2$=0.36 | $\lambda$=0.15<br>$\omega$=0.29<br>$\theta$=0.60 | $\lambda$=0.63<br>$\omega$=0.26<br>$\theta_1$=0.22<br>$\theta_2$=0.55<br>$\theta_3$=0.025 |
| $\chi^2$*-value* | | 880.21 | 880.21 | 188.64 | 297.9 | 418.01 | 7.48 |
| *P-value* | | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.2787 |
| *AIC* | | 1733.5 | 3344.1 | 910.6 | 3766.18 | 1220.5 | 840.25 |
| *BIC* | | 1734.3 | 3346.9 | 911.8 | 3769.9 | 1224.3 | 841.25.5 |
| *AICc* | | 1739.6 | 3345.8 | 915.3 | 3769.4 | 1224.6 | 845.7 |

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